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OSMUND JENSSEN

Shallow Hyperbolic Paraboloidal Shells

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Trondheim 1961

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Aktietrykkeriet i Trondhjem

SHALLOW HYPERBOLIC PARABOLOIDAL SHELLS

BY

OSMUND JENSSEN *)

SUMMARY

This paper deals with the bending theory of shallow hyperbolic paraboloid shell roofs. The solution is obtained in terms of power series, which converges for extremely shallow shells. The method to obtain the solution in the divergent domain is indicated as well.

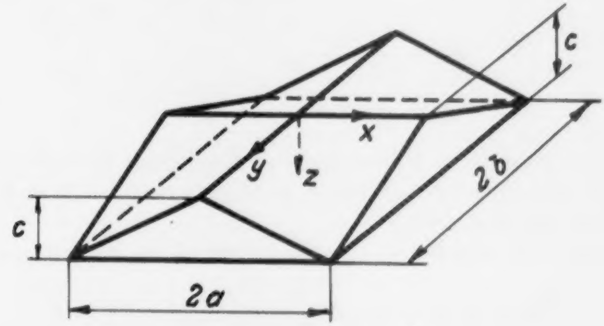

Fig. 1.

INTRODUCTION

In this paper the bending theory of Marguerre [1] for shallow shells is applied to the shell roof shown in fig. 1. The two differential equations of this theory are reduced to one single complex equation. This equation is separable but the Levy-type solution is not too well suited for the involved boundary conditions of our problem. In this work the differential equation is transformed into an integral equation in the way shown by Vekua [2]. The solution is obtained in terms of power series. Further the boundary conditions are established and an iteration procedure to satisfy these conditions is discussed.

BASIC EQUATIONS

The unstrained middle surface of the shell in the first octant fig. 1 obeys the equation

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with

$$z = kxy \tag{1}$$

$$k = \frac{c}{ab} \tag{2}$$

For this surface Marguerres [1] equations for the displacement w in z direction and the stress function Φ are given by

$$D\,\nabla^4 w + 2k\,\frac{\partial^2 \Phi}{\partial x \partial y} = q_z \tag{3}$$

$$\frac{1}{C}\,\nabla^4 \Phi - 2k\,\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{4}$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\,\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4},$$

$$D = \frac{Eh^3}{12(1-\nu^2)}, \tag{5}$$

$$C = Eh,$$

and q_z is the load intensity in z-direction. C is the stiffness in tension multiplied by $(1-\nu^2)$ and D the stiffness in bending. E is Young's modulus, ν is Poisson's ratio and h is the thickness of the shell.

Multiplying eq (4) by an arbitrary constant k_0 and adding it to eq (3) give

$$\nabla^4[w + k_0\Phi] - 2k\,k_0 C\,\frac{\partial^2}{\partial x \partial y}\left[w - \frac{1}{k_0 DC}\,\Phi\right] = \frac{q_z}{D} \tag{6}$$

The expressions in the brackets will be equal if

$$k_0 = -\frac{1}{k_0 DC}$$

giving

$$k_0 = i\,\frac{1}{(DC)^{\frac{1}{2}}} = i\,\frac{[12(1-\nu^2)]^{\frac{1}{2}}}{Eh^2} \tag{7}$$

With this value of k_0 equation (6) is transformed into

$$\nabla^4 U - \varepsilon\,\frac{\partial^2 U}{\partial x \partial y} = \frac{q_z}{D} \tag{8}$$

where

$$U = w + \frac{i}{(DC)^{\frac{1}{2}}}\,\Phi \tag{9}$$

and,

$$\varepsilon = i2k\left(\frac{C}{D}\right)^{\frac{1}{2}} \tag{10}$$

Forces and moments are expressed by w , Φ and the load intensities q_x , q_y in x and y directions as follows

$$N_x = \frac{\partial^2 \Phi}{\partial y^2} - \int q_x dx$$

$$N_y = \frac{\partial^2 \Phi}{\partial x^2} - \int q_y dy$$

$$N_{xy} = - \frac{\partial^2 \Phi}{\partial x \partial y}$$

$$M_x = - D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \tag{11}$$

$$My = - D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = - D (1 - \nu) \frac{\partial^2 w}{\partial x \partial y}$$

$$Q_x = - D \frac{\partial}{\partial x} (\nabla^2 w)$$

$$Q_y = - D \frac{\partial}{\partial y} (\nabla^2 w)$$

$$R_x = Q_x + \frac{\partial M_{xy}}{\partial y} + \frac{\partial z}{\partial x} N_x + \frac{\partial z}{\partial y} N_{xy}$$

$$R_y = Q_y + \frac{\partial M_{xy}}{\partial x} + \frac{\partial z}{\partial y} N_y + \frac{\partial z}{\partial x} N_{xy}$$

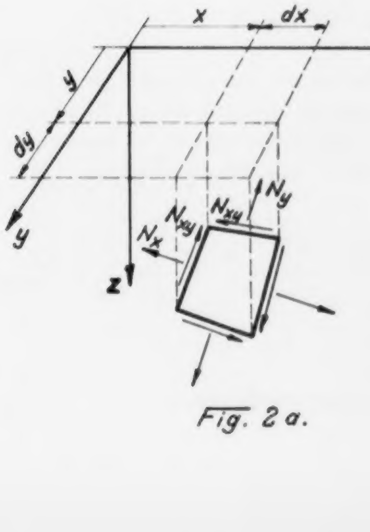

Fig. 2 a.

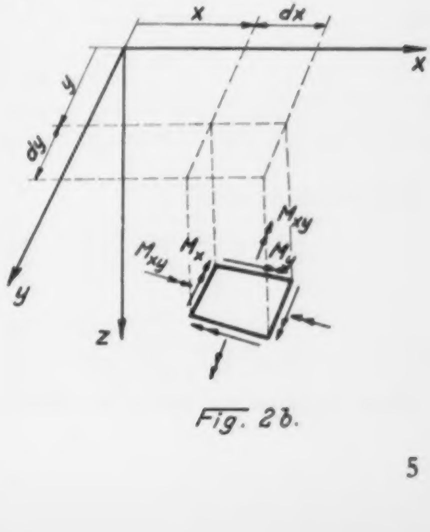

Fig. 2 b.

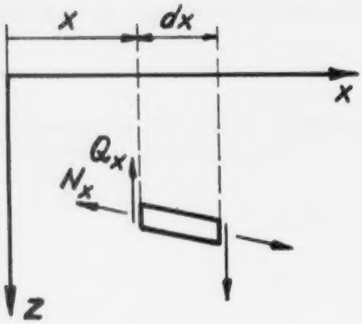

Fig. 2 c.

These forces and moments shown in fig. (2a–c) are applied to an element cut out from the shell by the planes x = const. x + dx = const., y = const. and y + dy = constant. N_x, N_{xy} and N_y are tangential to the middle surface, hence their vertical components are included in the effective vertical edge forces R_x and R_y.

The strains of the deformed middle surface are

$$
\begin{aligned}
\varepsilon_x &= \frac{1}{C}[N_x - \nu N_y] \\
\varepsilon_y &= \frac{1}{C}[N_y - \nu N_x] \\
\gamma &= \frac{2(1+\nu)}{C} N_{xy}
\end{aligned}
\tag{12}
$$

As shown by W. Flügge and F. T. Geyling [3] these strains may also be expressed in terms of the first derivatives of the displacement components u, v and w in x, y, z directions respectively as follows

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial w}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial y} \\
\gamma &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial w}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial w}{\partial y},
\end{aligned}
\tag{13a}
$$

from which we obtain

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial \gamma}{\partial y} - \frac{\partial \varepsilon_y}{\partial x} - \frac{\partial z}{\partial x}\frac{\partial^2 w}{\partial y^2} \\
\frac{\partial^2 v}{\partial x^2} &= \frac{\partial \gamma}{\partial x} - \frac{\partial \varepsilon_x}{\partial y} - \frac{\partial z}{\partial y}\frac{\partial^2 w}{\partial x^2}
\end{aligned}
\tag{13b}
$$

These equations are needed to establish our boundary conditions.

INTEGRATION OF THE HOMOGENEOUS DIFFERENTIAL EQUATION

In the equation to be solved, namely

$$\nabla^4 U_h - \varepsilon \frac{\partial^2 U_h}{\partial x \partial y} = 0\,, \tag{14}$$

where index h indicates that we deal with the homogeneous equation, we introduce the new complex and dimensionless variables

$$\begin{aligned} \zeta &= \xi + i\eta \\ \bar{\zeta} &= \xi - i\eta \end{aligned} \tag{15}$$

where

$$\xi = \frac{x}{a} \text{ and } \eta = \frac{y}{a} \tag{16}$$

and obtain the equation

$$\frac{\partial^4 U_h}{\partial \zeta^2 \partial \bar{\zeta}^2} - \frac{i\varepsilon a^2}{16}\left[\frac{\partial^2 U_h}{\partial \zeta^2} - \frac{\partial^2 U_h}{\partial \bar{\zeta}^2}\right] = 0 \tag{17}$$

By integration this equation takes the form

$$U_h = -\varkappa\left[\int\!\!\int U_h d\zeta d\zeta - \int\!\!\int U_h d\bar{\zeta} d\bar{\zeta}\right] \tag{18}$$

with

$$\varkappa = \frac{ca}{8b}\left(\frac{C}{D}\right)^{\frac{1}{2}} = \frac{1}{8}\frac{a}{b}\frac{c}{h}\sqrt{12(1-\nu^2)} \tag{19}$$

as easily obtained from eq (17) by use of eq (2) (10) and (5). For U_h in the right side of the eq (18) we now substitute the biharmonic polynoms

$$U_{hn} = A_n\zeta^n + B_n\bar{\zeta}^n + C_n\bar{\zeta}\zeta^{n-1} + D_n\zeta\bar{\zeta}^{n-1} \tag{20}$$

where A_n, B_n, C_n and D_n are arbitrary constants.

Let U^1_{hn} be the function obtained in this way on the left side of eq. (18). Next we substitute U^1_{hn} on the right side of eq (18) and obtain on the left side the new function U^2_{hn}. Going on in this way we find

$$U_h = \sum_{n=2}^{\infty}\sum_{q=0}^{\infty} U^q_{hn} = \sum_{n=2}^{\infty}\sum_{q=0}^{\infty} \varkappa^q \left\{\frac{n!}{(n+2q)!}\sum_{\alpha=0}^{q}(-1)^\alpha \binom{q}{\alpha}\binom{n+2q}{2\alpha} P_{nq\alpha} + \right.$$

$$\left. \frac{(n-1)!}{(n+2q)!}\sum^{q}(-1)^\beta\binom{q}{\beta}\binom{n+2q}{2\beta+1} H_{nq\beta}\right\} \tag{22}$$

where

$$P_{nq_\alpha} = [A_n\zeta^{n+2q-4\alpha} + (-1)^q B_n \xi^{n+2q-4\alpha}](\zeta\xi)^{2\alpha} \tag{23}$$

and

$$H_{nq_\beta} = [C_n\zeta^{n-2+2q-4\beta} + (-1)^q D_n \xi^{n-2+2q-4\beta}](\zeta\xi)^{2\beta+1}$$

The polynoms U^q_{hn} in eq (22) should satisfy the equation

$$\frac{\partial^4 U^q_{hn}}{\partial\zeta^2\partial\xi^2} + \varkappa\left[\frac{\partial^2 U^{q-1}_{hn}}{\partial\zeta^2} - \frac{\partial^2 U^{q-1}_{hn}}{\partial\xi^2}\right] = 0 \tag{24}$$

which can be verified by direct substitution.

It is often convenient to work with solution wich are symmetric or antisymmetric with respect to ξ and η. These are obtained by proper choice of the constants of integration as given in the following table.

n	Symmetric solution		Antisymmetric solution	
$n = 4j$	$A_n = B_n = \frac{1}{2}a_n$	$-C_n = D_n = i\frac{1}{2}c_n$	$-A_n = B_n = i\frac{1}{2}b_n$	$C_n = D_n = \frac{1}{2}d_n$
$n = 4j + 2$	$-A_n = B_n = i\frac{1}{2}a_n$	$C_n = D_n = \frac{1}{2}c_n$	$A_n = B_n = \frac{1}{2}b_n$	$-C_n = D_n = i\frac{1}{2}d_n$
$n = 4j + 1$	$A_n = \frac{(1-i)}{2}a_n$	$C_n = \frac{(1+i)}{2}c_n$	$A_n = \frac{(1+i)}{2}b_n$	$C_n = \frac{(1-i)}{2}d_n$
	$B_n = \frac{(1+i)}{2}a_n$	$D_n = \frac{(1-i)}{2}c_n$	$B_n = \frac{(1-i)}{2}b_n$	$D_n = \frac{(1+i)}{2}d_n$
$n = 4j + 3$	$A_n = \frac{(1+i)}{2}a_n$	$C_n = \frac{(1-i)}{2}c_n$	$A_n = \frac{(1-i)}{2}b_n$	$C_n = \frac{(1+i)}{2}d_n$
	$B_n = \frac{(1-i)}{2}a_n$	$D_n = \frac{(1+i)}{2}c_n$	$B_n = \frac{(1+i)}{2}b_n$	$D_n = \frac{(1-i)}{2}d_n$

where n, and j are integers and $a_n - d_n$ are new complex constants of integration. The solution given by eq (22) and (23) is independent of the choice of origin, hence we may add the solution obtained by putting the origin at the point $\xi = 1$ and $\eta = 1$, which corresponds to the substitution of $\xi - 1$ and $\eta - 1$ for ξ and η in eq (22) and (23). Let the constants of integration connected with this solution be denoted by $a'_n - d'_n$.

INTEGRATION OF THE NON-HOMOGENEOUS EQUATION

The considered equation namely eq (8) may by use of eq (16), (10) and (19) be written

$$\frac{\partial^4 U}{\partial \xi^4} + 2\frac{\partial^4 U}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 U}{\partial \eta^4} - i\varkappa 16 \frac{\partial^2 U}{\partial \xi \partial \eta} = \frac{a^4 q_z}{D} \tag{25}$$

When U is a smooth function and $\varkappa$ a large parameter, we can neglect the fourth derivatives in this equation and obtain

$$\frac{\partial^2 U_p}{\partial \xi \partial \eta} = i\frac{a^4}{16 D\varkappa} q_z \tag{26}$$

where index p indicates that we deal with the particular solution. For $q_z = q_0 =$ const. we obtain from eq (26), (9), (11) and (19) the following well known membrane solution

$$(N_{xy})_p = -\frac{\partial^2 \Phi_p}{\partial x \partial y} = -\frac{1}{a^2}\frac{\partial^2 \Phi_p}{\partial \xi \partial \eta} = -\frac{ab}{2c} q_0 \tag{27}$$

In this case and when q_z is a linear function of ξ and η eq (25) is identically satisfied. For other loading functions and smaller values of $\varkappa$ the solution can be found by the method of successive approximations.

BOUNDARY CONDITIONS

We consider the shell shown in fig. 1 with $q_x = q_y = 0$ and $q_z = q_0 =$ const. applied to the entire roof. The edge stiffening members for $x = a$ and $y = b$ are par example assumed to be rigid in the direction of their axes and have negligible bending resistance in planes tangent to the shell. Hence w , the edge moment, N_x and N_y should vanish along the outside boundary giving

$$\left.\begin{aligned} \xi &= 1 \\ \eta &= \frac{b}{a} \end{aligned}\right\} w = \nabla^2 w = \Phi = \nabla^2 \Phi = 0$$

which can be written as

$$\left.\begin{aligned} \xi &= 1 \\ \eta &= \frac{b}{a} \end{aligned}\right\} U = \nabla^2 U = 0 \tag{28}$$

For $y = 0$ we have from symmetry the conditions

$$\frac{\partial w}{\partial y} = 0 \tag{29}$$

and

$$\frac{\partial^2 v}{\partial x^2} = 0 \text{ together with } \gamma_{x=y=0} = 0$$

which according to eq (13), (12), (1) and (2) can be written

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{C}\left[2(1+\nu)\frac{\partial N_{xy}}{\partial x} - \frac{\partial N_x}{\partial y} + \nu\frac{\partial N_y}{\partial y}\right] - kx\frac{\partial^2 w}{\partial x^2} = 0 \tag{30a}$$

and

$$(N_{xy})_{x=y=0} = 0 \tag{30b}$$

This edge has to be stiffened by an edge beam with area and moment of inertia denoted F_ξ and I_ξ. Hence we have to add the two conditions

$$(\varepsilon_x)_{y=0} = \varepsilon_0 \tag{31}$$

and

$$\left(\frac{\partial^2 w}{\partial x^2}\right)_{y=0} = \frac{d^2 w_0}{dx^2} \tag{32}$$

where the terms with index zero are the strain and curvature of the edge beam. We now assume that the neutral planes of the edge beam and the shell coinside at the line $y = \eta = 0$. Hence the N_{xy} forces on both sides of the edge beam give

$$\varepsilon_0 = \int_x^a \frac{2N_{xy}}{EF_\xi}\,d\tau$$

and equation (31) takes by use of eq. (12) the form

$$(\varepsilon_x)_{y=0} = \frac{1}{C}[N_x - \nu N_y]_{y=0} = \int_x^a \frac{2N_{xy}}{EF_\xi}\,d\tau \tag{33}$$

The vertical load pr. unit length of the edge beam consists of its own weight and external load denoted by g_ξ and the effective edge force R_y on both sides of the edge beam as shown on fig. 3. This gives the shearing force

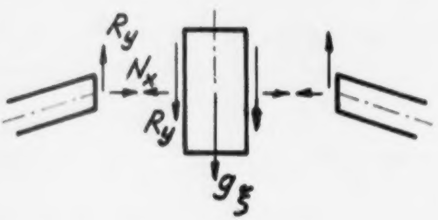

Fig. 3.

$$Q = -\int_0^x (2R_y + g_\xi)\,d\tau \tag{34}$$

and the bending moment

$$M = -\int_x^a Q\,d\varrho = \int_x^a d\varrho \int_0^\varrho (2R_y + g_\xi)\,dx \tag{35}$$

which determines the curvature of the beam given by the equation

$$\frac{d^2 w_0}{dx^2} = -\frac{M}{EI_\xi} \tag{36}$$

Eq (32) may then by use of eq (34–36) be written

$$EI_\xi\left(\frac{\partial^2 w}{\partial x^2}\right)_{y=0} + \int_x^a d\varrho \int_0^\varrho (2R_y + g_\xi)dx = 0 \tag{37}$$

Our boundary conditions at the edge $y = \eta = 0$ given by eq (29), (30a), (33), and (37) can now be given in terms of U by use of the equations (11), (9), (1), (16) and (19) as follows

$$\eta = 0 \left\{ \begin{aligned} & \mathrm{Re}\left(\frac{\partial U}{\partial \eta}\right) = 0 \\ & \mathrm{Im}\left[(2+\nu)\frac{\partial^3 U}{\partial \xi^2 \partial \eta} + \frac{\partial^3 U}{\partial \eta^3}\right] + 8\varkappa \mathrm{Re}\left[\xi \frac{\partial^2 U}{\partial \xi^2}\right] = 0 \\ & \mathrm{Im}\left[\frac{\partial^2 U}{\partial \eta^2} - \nu \frac{\partial^2 U}{\partial \xi^2}\right] + \gamma_\xi \mathrm{Im}\left[\left(\frac{\partial U}{\partial \eta}\right)_{\xi=1} - \frac{\partial U}{\partial \eta}\right] = 0 \\ & \mathrm{Re}\left[a\left(\frac{\partial^2 U}{\partial \xi^2}\right) - \int_\xi^1 d\varrho \int_0^\varrho \left[(2-\nu)\frac{\partial^3 U}{\partial \xi^2 \partial \eta} + \frac{\partial^3 U}{\partial \eta^3}\right] d\xi\right] + \\ & \qquad 8\varkappa \mathrm{Im}\left[\int_\xi^1 d\varrho \int_0^\varrho \xi \frac{\partial^2 U}{\partial \xi^2} d\xi\right] + \beta_\xi(1 - \xi^2) = 0 \end{aligned} \right. \tag{38}$$

where Re [ ] and Im [ ] denotes the real and imaginary part of the term in the brackets respectively and

$$a_\xi = \frac{EI_\xi}{2Da}, \quad \beta_\xi = \frac{a^3 g_\xi}{4D}, \quad \gamma_\xi = \frac{2Ca}{EF_\xi} \tag{39}$$

The similar expressions for the edge $\xi = 0$ are obtained by substitution of η for ξ and ξ for η in eq (38) and (39).

These equations have to be supplemented with eq (30b) which can be written

$$\mathrm{Im}\left(\frac{\partial^2 U}{\partial \xi \partial \eta}\right)_{\xi=\eta=0} = 0 \tag{40}$$

The boundary conditions are expressed by the derivatives of U with respect to ξ and η while the homogeneous solution is expressed in terms of ζ and $\bar{\zeta}$, hence we list the following transforms where $m = n + 2q - 4a$

$$\frac{\partial}{\partial \xi}(P_{nq\alpha}) = \left[\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \bar{\zeta}}\right] P_{nq\alpha}$$

$$= (n + 2q - 2\alpha)\,[A_n\zeta^{m-1} + (-1)^q B_n\bar{\zeta}^{m-1}]\,(\zeta\bar{\zeta})^{2\alpha}$$

$$+ (2\alpha)\,[A_n\zeta^{m+1} + (-1)^q B_n\bar{\zeta}^{m+1}](\zeta\bar{\zeta})^{2\alpha-1}$$

$$\frac{\partial}{\partial \eta}(P_{nq\alpha}) = i\left[\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \bar{\zeta}}\right] P_{nq\alpha}$$

$$= i\,\{(n + 2q - 2\alpha)\,[A_n\zeta^{m-1} + (-1)^{q+1} B_n\bar{\zeta}^{m-1}]\,(\zeta\bar{\zeta})^{2\alpha}$$

$$- (2\alpha)\,[A_n\zeta^{m+1} + (-1)^{q+1} B_n\bar{\zeta}^{m+1}]\,(\zeta\bar{\zeta})^{2\alpha-1}\}$$

$$\frac{\partial^2}{\partial \xi^2}(P_{nq\alpha}) = \left[\frac{\partial^2}{\partial \zeta^2} + 2\,\frac{\partial^2}{\partial \zeta\,\partial \bar{\zeta}} + \frac{\partial^2}{\partial \bar{\zeta}^2}\right] P_{nq\alpha}$$

$$= (n + 2q - 2\alpha)(n + 2q - 2\alpha - 1)\,[A_n\zeta^{m-2} + (-1)^q B_n\bar{\zeta}^{m-2}]\,(\zeta\bar{\zeta})^{2\alpha}$$

$$+ 2(2\alpha)(n + 2q - 2\alpha)\,[A_n\zeta^{m} + (-1)^q B_n\bar{\zeta}^{m}]\,(\zeta\bar{\zeta})^{2\alpha-1}$$

$$+ (2\alpha)\,(2\alpha - 1)\,[A_n\zeta^{m+2} + (-1)^q B_n\bar{\zeta}^{m+2}]\,(\zeta\bar{\zeta})^{2\alpha-2}$$

$$\frac{\partial^2}{\partial \eta^2}(P_{nq\alpha}) = \left[-\frac{\partial^2}{\partial \zeta^2} + 2\,\frac{\partial^2}{\partial \zeta\partial \bar{\zeta}} - \frac{\partial^2}{\partial \bar{\zeta}^2}\right] P_{nq\alpha}$$

$$= -(n + 2q - 2\alpha)\,(n + 2q - 2\alpha - 1)\,[A_n\zeta^{m-2} + (-1)^q B_n\bar{\zeta}^{m-2}](\zeta\bar{\zeta})^{2\alpha}$$

$$+ 2(2\alpha)(n + 2q - 2\alpha)\,[A_n\zeta^{m} + (-1)^q B_n\bar{\zeta}^{m}]\,(\zeta\bar{\zeta})^{2\alpha-1}$$

$$- (2\alpha)(2\alpha - 1)\,[A_n\zeta^{m+2} + (-1)^q B_n\bar{\zeta}^{m+2}]\,(\zeta\bar{\zeta})^{2\alpha-2}$$

$$\frac{\partial^2}{\partial \xi\partial \eta}\,(P_{nq\alpha}) = i\left[\frac{\partial^2}{\partial \zeta^2} - \frac{\partial^2}{\partial \bar{\zeta}^2}\right] P_{nq\alpha}$$

$$= i\,\{(n + 2q - 2\alpha)(n + 2q - 2\alpha - 1)\,[A_n\zeta^{m-2} + (-1)^{q+1} B_n\bar{\zeta}^{\,m-2}](\zeta\bar{\zeta})^{2\alpha}$$

$$- (2\alpha)(2\alpha - 1)[A_n\zeta^{m+2} + (-1)^{q+1} B_n\bar{\zeta}^{m+2}]\,(\zeta\bar{\zeta})^{2\alpha-2}\}$$

$$\left[(2 \pm \nu)\frac{\partial^3}{\partial \xi^2\partial \eta} + \frac{\partial^3}{\partial \eta^3}\right] P_{nq\alpha} = i\left[(1 \pm \nu)\,\frac{\partial^3}{\partial \zeta^3} + (5 \pm \nu)\,\frac{\partial^3}{\partial \zeta^2\partial \bar{\zeta}} - (5 \pm \nu)\,\frac{\partial^3}{\partial \zeta\partial \bar{\zeta}^2}\right.$$

$$\left. - (1 \pm \nu)\,\frac{\partial^3}{\partial \bar{\zeta}^3}\right] P_{nq\alpha}$$

which in turn is eqnal to the expression on the following page

$$= i\,\{(1 \pm \nu)(n + 2q - 2\alpha)(n + 2q - 2\alpha - 1)(n + 2q - 2\alpha - 2)\,[A_n\zeta^{m-3} + (-1)^{q+1}B_n\xi^{m-3}]\,(\zeta\xi)^{2\alpha}$$

$$+ (5 \pm \nu)(2\alpha)(n + 2q - 2\alpha)(n + 2q - 2\alpha - 1)\,[A_n\zeta^{m-1} + (-1)^{q+1}B_n\xi^{m-1}](\zeta\xi)^{2\alpha-1}$$

$$- (5 \pm \nu)(2\alpha)(2\alpha - 1)(n + 2q - 2\alpha)\,[A_n\zeta^{m+1} + (-1)^{q+1}B_n\xi^{m+1}]\,(\zeta\xi)^{2\alpha-2}$$

$$- (1 \pm \nu)(2\alpha)(2\alpha - 1)(2\alpha - 2)\,[A_n\zeta^{m+3} + (-1)^{q+1}B_n\xi^{m+3}]\,(\zeta\xi)^{2\alpha-3}\}$$

The similar expressions for the polynoms $H_{nq\beta}$ are obtained by substitution of $H_{nq\beta}$ for $P_{nq\alpha}$, C_n and D_n for A_n and B_n and $2\beta + 1$ for 2α.

APPLICATION OF THE SOLUTION

We take as an example a shell with $a = b$, $\alpha_\xi = \alpha_\eta = \alpha$, $\beta_\xi = \beta_\eta = \beta$, $\gamma_\xi = \gamma_\eta = \gamma$, $\nu = 0$ and with uniformly distributed load $q_z = q_0$ applied to the entire roof (fig. 1); hence we need only the symmetrical part of the solution eq. (22).

We have to calculate two sets of constants namely a_n, c_n and a_n', c_n' corresponding to the solutions U and U′ obtained when the origin of coordinates in eq (22) is placed at $\xi = \eta = 0$ and $\xi = \eta = 1$ respectively. These constants can be calculated from the equations obtained by equating to zero the coefficients of ξ^n for each n at the line $\eta = 0$ and of η^n where $\xi = 1$. This infinite set of equations will not be easy to solve. However, by neglecting the influence of the solution U′ at the boundary $\eta = 0$ we obtain a set of equations from which the constants a_n and c_n are easily found. This solution gives disturbances at the boundaries $\xi = 1$ and $\eta = 1$ which can be eliminated by addition of an U′ solution. This gives disturbances at the boundaries $\xi = 0$ and $\eta = 0$ which may be eliminated by a new U solution, which again has influence at the outer boundaries, and so on.

The convergence of this iteration procedure will depend on the N_x and N_y forces on the boundaries $\xi = 1$ and $\eta = 1$ obtained from the first U solution. The first U′ solution will then include equal and opposite N_x and N_y forces on these boundaries and when these forces have a resultant there will be a beam action between the parallel boundaries. This beam action is one of the main causes for the reflected disturbances on the inner boundaries. The more evenly distributed these N_x and N_y forces is the more undamped will they go from one edge to the opposite with correspondingly bad convergence of this iteration procedure. Let us now consider the first step namely the calculation of the constants a_n, c_n from the conditions at the boundary $\eta = 0$ given by the equations (38). These conditions result in the following set of equations for the real and imaginary part of the coefficients denoted by subscripts r and i respectively

$$a_{2r} = 0$$

$$3a_{3r} - c_{3r} = 0$$

$$4c_{4r} - \frac{4}{3} \varkappa\, c_{2i} = 0$$

$$10a_{5r} - 6c_{5r} + \frac{5}{2} \varkappa a_{3i} - \frac{1}{2} \varkappa c_{3i} = 0 \tag{42}$$

$$12a_{6r} + \frac{8}{5} \varkappa a_{4i} - \frac{7}{15} \varkappa^2 a_{2r} = 0$$

— — — —

$$-6a_{3i} + 10c_{3i} = 0$$

$$36c_{4i} + 28\varkappa c_{2r} = 0$$

$$60a_{5i} - 84c_{5i} + 48\varkappa(a_{3r} + c_{3r}) = 0$$

$$120a_{6i} + 48\varkappa a_{4r} - \frac{158}{3} \varkappa^2 a_{2i} \tag{43}$$

— — — —

$$2c_{2i} + \gamma \operatorname{Im} \left(\frac{\partial U}{\partial \eta}\right)_{\xi=1} = 4\gamma a_{2i}$$

$$-3a_{3i} + c_{3i} = 0$$

$$-6a_{4i} - \frac{1}{2} \varkappa a_{2r} + \gamma(3a_{3i} - c_{3i}) = 0 \tag{44}$$

$$20a_{5i} + 4c_{5i} + 3\varkappa a_{3r} - \varkappa c_{3r} + \gamma \left(4c_{4i} + \frac{4}{3} \varkappa c_{2r}\right) = 0$$

$$-10c_{6i} + \frac{3}{2} \varkappa c_{4r} - \frac{1}{2} \varkappa^2 c_{2i} - \gamma \left(5a_{5i} - 3c_{5i} - \frac{3}{4} \varkappa a_{3i} + \frac{5}{12} \varkappa c_{3i}\right) = 0$$

— — — —

$$2\alpha c_{2r} - \operatorname{Re} \left[\int_0^1 d\varrho \int_0^\varrho \left(2 \frac{\partial^3 U}{\partial \xi^2 \partial \eta} + \frac{\partial^3 U}{\partial \eta^3}\right) d\xi\right] +$$

$$+ 8\varkappa \operatorname{Im} \left[\int_0^1 d\varrho \int_0^\varrho \xi \frac{\partial^2 U}{\partial \xi^2}\, d\xi\right] + \beta = 0$$

$$
\begin{aligned}
&a_{3r} + c_{3r} = 0 \\
&\alpha(12a_{4r} - 5\varkappa a_{21}) - \frac{1}{2}(-3a_{3r} + 5c_{3r}) - \beta = 0 \\
&\alpha(20a_{5r} + 20c_{5r} + 9\varkappa a_{31} - \frac{5}{3}\varkappa c_{31}) + \frac{1}{2\cdot 3}(36c_{4r} - 28\varkappa c_{21}) = 0 \\
&\alpha\left(30c_{6r} - \frac{7}{2}\varkappa c_{41} - \frac{7}{6}\varkappa^2 c_{2r}\right) + \frac{1}{3\cdot 4}(60a_{5r} - 84c_{5r} + 63\varkappa a_{31} - 29\varkappa c_{31}) = 0 \\
&- \quad - \quad - \quad -
\end{aligned}
\tag{45}
$$

From these four sets of infinite equations all the constants with subscripts of higher number than two can be expressed by a_{21}, c_{2r} and c_{21}, and these constants can then be determined from the first of eqs (44) and (45) together with the boundary condition (40). By aid of (27) this can be written

$$
\mathrm{Im}\left(\frac{\partial^2 U_h}{\partial\xi\partial\eta}\right)_{\xi=\eta=0} + \frac{a^4}{16\varkappa D} q_0 = 0
$$

and by use of eq (22), (41) and (19) we obtain

$$
a_{21} = -\frac{a^4 q_0}{32\varkappa D} = -2\varkappa\frac{a^4}{c^2 h}\frac{q_0}{E} \tag{46}
$$

CONVERGENCE OF THE SERIES

For large values of n and j the terms with the largest subscripts in the equations (42–45) will dominate and the constants may be calculated as follows.

We want to calculate par example $a_{(4j+2)i}$. This can be done by aid of the boundary conditions (38b) and (38d). By equating to zero the coefficient of ξ^{4j-1} the first of the above mentioned conditions gives

$$
\begin{aligned}
&(4j)^3\left\{a_{(4j+2)i}\left[1 + 0\left(\frac{1}{4j}\right)\right] - \varkappa a_{(4j)r}\left[\frac{1}{2!} + 0\left(\frac{1}{4j}\right)\right] + \varkappa^2 a_{(4j-2)i}\left[\frac{1}{4!} + 0\left(\frac{1}{4j}\right)\right]\right. \\
&\qquad\qquad \left. - \varkappa^3 a_{(4j-4)r}\left[\frac{1}{6!} + 0\left(\frac{1}{4j}\right)\right] \ldots\ldots\ldots\ldots\right\} \\
&\quad + 8\varkappa(4j)^2\left\{a_{(4j)r}\left[1 + 0\left(\frac{1}{4j}\right)\right] - \varkappa a_{(4j-2)i}\left[\frac{1}{2!} + 0\left(\frac{1}{4j}\right)\right]\ldots\right\} = 0
\end{aligned}
\tag{47}
$$

which has been developed by use of eq (22) and (41). When j is very large we may neglect the terms of order of magnitude $\frac{1}{4j}$ and eq (47) results in

$$a_{(4j+2)i} = \frac{\varkappa}{2!} a_{(4j)r} - \frac{\varkappa^2}{4!} a_{(4j-2)i} + \frac{\varkappa^3}{6!} a_{(4j-4)r} \cdot\cdot \tag{48}$$

In eq (38d) the first term is proportional to $(4j)^2$ while the next two terms are proportional to (4j) and $8\varkappa$ respectively, hence this boundary condition may for sufficiently large values of j be written as

$$\mathrm{Re}\left(\frac{\partial^2 U}{\partial \xi^2}\right) = 0 \tag{49}$$

Equating to zero the coefficient of ξ^{4j-2} in this condition we get by neglecting all terms of order of magnitude $\frac{1}{4j}$ the equation

$$a_{(4j)r} = \frac{\varkappa}{2!} a_{(4j-2)i} - \frac{\varkappa^2}{4!} a_{(4j-4)r} + \frac{\varkappa^3}{6!} a_{(4j-6)i} \cdots \tag{50}$$

We now substitute this value of $a_{(4j)r}$ in eq (48) and obtain

$$a_{(4j+2)i} = \varkappa^2 \left[\left(\frac{1}{2!}\right)^2 - \frac{1}{4!}\right] a_{(4j-2)i} + \varkappa^3 \left[-\frac{1}{2!4!} + \frac{1}{6!}\right] a_{(4j-4)r} + \cdot\cdot \tag{51}$$

In accordance with eq (48) we obtain

$$a_{(4j-4)r} \sim \frac{2!}{\varkappa} a_{(4j-2)i} \tag{52}$$

which substituted in eq (51) gives

$$\frac{a_{(4j+2)i}}{a_{(4j-2)i}} \sim \left[\left(\frac{1}{2!}\right)^2 - \frac{2}{4!} + \frac{2!}{6!}\right] \varkappa^2 \tag{53}$$

In the same way eq (50) and (48) result in

$$\frac{a_{(4j)r}}{a_{(4j-4)r}} \sim \left[\left(\frac{1}{2!}\right)^2 - \frac{2}{4!} - \frac{2!}{6!}\right] \varkappa^2 \tag{54}$$

For other coefficients we obtain similar expressions, where the main term on the right side will be either $\left(\frac{\varkappa}{2!}\right)^2$ as in eq (53) and (54) or $\left(\frac{\varkappa}{3!}\right)^2$. The series seem therefore to be convergent when $\varkappa < 2$ and divergent for $\varkappa > 2$. However, no singularity in the solution should be expected within or on the boundary of the shell, hence one of the methods of summing divergent series may be used. For example transformation into continued fractions after Ruthishausers «Q. D. algorithmus» [4].

To calculate c_{2r} and c_{2i} we want the value of the function $\mathrm{Im}\left(\frac{\partial U}{\partial \eta}\right)_{\eta=0}$ where $\xi = 1$.

The contribution to this function from the coefficients of ξ^{4j+1} can be written

$$\sum_{j=0}^{\infty}(4j+2)\xi^{4j+1}\sum_{p=0}^{2j}\frac{(-\varkappa)^p}{2p!}a_{(4j+2-2p)s}\left[1-\frac{p}{j}-\frac{2p(2p^2-p-4)}{(4j)^2}+0\left(\frac{p^3}{j^3}\right)\right] \tag{55}$$

where the index s in $a_{(4j+2-2p)s}$ is equal to i, when p is an even number and equal to r when p is an odd number. The series given by (55) will according to eq (53) and (54) for a given value of ξ result in a function with a singularity for a sufficiently large $\varkappa$-value. Hence, the contributions from the different types of coefficients can not be given separately and we must express all the different coefficients by external loads, that is by a_{2i} and β, before the series expansions of the different derivatives of the solution can be summed. This however is impossible without knowing the values of c_{2i} and c_{2r}. These coefficients may for $\varkappa > 2$ be found in the following way.

In the first of the equations (44) we substitute for $\operatorname{Im}\left(\frac{\partial U}{\partial \eta}\right)$ at the points $\xi = 1$, $\eta = 0$ the contributions from the different types of coefficients. In these expressions which are similar to (55) we change the upper limit of summation from infinity to a finite number denoted by m. Further we treat the first of the equations (45) in the same way and denote the values of c_{2r} and c_{2i} obtained from these equations $(c_{2r})_m$ and $(c_{2i})_m$. The values of c_{2r} and c_{2i} may then be found by the equations:

$$\begin{aligned} c_{2r} &= \lim_{N\to\infty}\left(\frac{1}{N}\sum_{m=1}^{N}(c_{2r})_m\right) \\ c_{2i} &= \lim_{N\to\infty}\left(\frac{1}{N}\sum_{m=1}^{N}(c_{2i})_m\right) \end{aligned} \tag{56}$$

FINAL REMARKS

By large $\varkappa$-values we should expect a zone of quickly damped disturbances in the neighborough of the origin $\xi = \eta = 0$. The N_x, N_y forces on the inner boundaries outside this zone might go nearly undisturbed to the outer boundaries with correspondingly bad convergence of the iteration procedure. It should be mentioned here that the nearly undamped solution may be somewhat incorrect, as the Donell solution for the boundary disturbances from the curved edge in cylindrical shells. This is shown by Holand [5]. Hence, the error in the solution by repeated reflection of bondary disturbances may be considerable. However, the N_x and N_y forces on the outer boundaries may be changed into shearing forces and moments by proper addition of solutions as the following

$$U = K\left[i\xi^a\eta - \frac{4\varkappa}{30}\xi^6\right] \tag{57}$$

where K is a real constant. From this equation and eq (11) we obtain with $\nu = 0$

$$
\begin{aligned}
N_x &= 0 \\
N_y &= 6K(DC)^{\frac{1}{2}}\xi\eta \\
N_{xy} &= -3K(DC)^{\frac{1}{2}}\xi^2 \\
M_x &= 4K\,D\xi^4 \\
M_y &= 0 \\
Q_x &= 16K\,D\xi^3 \\
Q_y &= 0
\end{aligned}
\tag{58}
$$

The N_x, N_y forces which have to be eliminated with solutions of the same kind as eq (57), originate from the load of the inner stiffening beams and may for small beams be negligible. However, for a shallow shell loaded with a large snow load we need for longer spans a large edge beam to transmit the compressive forces. The cross section can par example be made as shown in fig. 4. If the stiffness of the edge beam in the transverse direction is large compared to the stiffness of the shell and the width is small compared to the length, we can still use the boundary conditions eq (38).

Considering now the shells with a $\varkappa$-value less than 2. For a shell with a = b and $\nu = 0$ we obtain from eq (19)

$$\frac{c}{h} < 4{,}62 \tag{59}$$

Fig. 4.

which corresponds to very shallow shells with comparatively short spans. However, if hollow sections (fig. 5) are used, shells with $\varkappa < 2$ covering great areas may be constructed. One may realize this by rewriting eq (19) as

$$\varkappa = \frac{c}{8r}\frac{a}{b}\sqrt{1-\nu^2} \tag{60}$$

where r is the radius of gyration of the cross section.

Fig. 5.

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